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Isothermal-Conjugate Systems of Lines on Surfaces.

By L. P. Eisenhart.

Bianchi* has given the name *isothermal-conjugate* to a double system of lines, on a surface of positive curvature, which are such that when the surface is referred to them as parametric lines, the second fundamental quadratic form for the surface can be brought to the form

$$\lambda(du^2+dv^2).$$

We have extended the term so that it refers also to conjugate systems on surfaces of negative curvature; in this case the lines must be such that the second fundamental form can be written

$$\lambda(du^2-dv^2),$$

when the lines $u=\text{const.}$, $v=\text{const.}$ form an isothermal-conjugate system. It is in this broader sense that we shall treat of such systems.

In the first section we show how these lines can be determined, and by methods similar to those used by Bianchi† in his treatment of isothermal orthogonal systems, we establish theorems similar to those which are well known in the theory of these latter systems. And for surfaces of negative curvature we establish results very similar to those which Bianchi‡ has found for surfaces of positive curvature. A geometrical interpretation of these systems leads to the theorem that the surfaces, obtained by reciprocal radii vectores from surfaces whose lines of curvature are isothermal-conjugate, are of the same kind.

In the second section, we begin the study of surfaces whose lines of curvature form an isothermal-conjugate system, and show that the general problem of determining these surfaces depends upon the integration of a differential equation of the fourth order, very similar to the equation found by Darboux in his study of isothermic surfaces. After showing that the surfaces whose lines of curvature are at the same time isothermal and isothermal-conjugate have an isothermal spherical representation, we pass to the theorem that the sphere, certain surfaces of revolution and minimal surfaces are the only surfaces of constant mean curvature whose lines of curvature are isothermal-conjugate.

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* Lezioni, p. 132.

† Ib., p. 67.

‡ Bianchi, p. 133.

By the direct calculation of the coefficients of their second fundamental forms, we show that quadric surfaces and surfaces of revolution have isothermal-conjugate lines of curvature.

By means of a method similar to that used by Willgrod,[*] we solve the problem of finding all surfaces which, together with their parallels, have isothermal-conjugate lines of curvature We find that the sphere, plane, cyclides of Dupin and surfaces of revolution, are the only surfaces of this kind, and we note further that these surfaces are isothermic.

In addition to the surfaces of constant total curvature and surfaces of revolution, the surfaces whose radii satisfy the conditions

$$\frac{1}{\rho_1\rho_2}=f(U+V),\quad \frac{\rho_2-\rho_1}{\rho_1\rho_2(\rho_1+\rho_2)}=f'(U+V),$$

where f, U, V are bound by a certain relation, have isothermal-conjugate lines of curvature.

The paper closes with a discussion of surfaces with plane lines of curvature in both systems and, at the same time, isothermal-conjugate. Solutions of this problem are furnished by surfaces of revolution, certain moulure surfaces, cyclides of Dupin, minimal surfaces of Bonnet and Enneper and other surfaces whose coordinates are given.

I.

Consider a surface (or region of a surface) of *positive* total curvature. The quadratic differential form

$$f=Ddu^2+2D'du\,dv+D''dv^2, \tag{1}$$

which, when equated to zero, gives the imaginary asymptotic directions at the corresponding point, can be decomposed into two conjugate imaginary factors, thus:

$$f=\left(\sqrt{D}du+\frac{D'+i\sqrt{DD''-D'^2}}{\sqrt{D}}dv\right)\left(\sqrt{D}du+\frac{D'-i\sqrt{DD''-D'^2}}{\sqrt{D}}dv\right). \tag{2}$$

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[*] "Ueber Flächen, welche sich durch ihre Krummungslinien in unendlich kleine Quadrate theilen lassen (Dissertation, Göttingen, 1883).

From integral calculus we know that there is an infinity of integrating factors of

$$\sqrt{D}\,du + \frac{D' + i\sqrt{DD'' - D'^2}}{\sqrt{D}}\,dv; \tag{3}$$

let one of them be $\mu + i\nu$, and let $\phi + i\psi$ be the function of which the product of $\mu + i\nu$ and (3) is the exact differential, then

$$(\mu + i\nu)\left(\sqrt{D}\,du + \frac{D' + i\sqrt{DD'' - D'^2}}{\sqrt{D}}\,dv\right) = d(\phi + i\psi), \tag{4}$$

and

$$(\mu - i\nu)\left(\sqrt{D}\,du + \frac{D' - i\sqrt{DD'' - D'^2}}{\sqrt{D}}\,dv\right) = d(\phi - i\psi). \tag{5}$$

If, now, the surface is referred to the system of lines $\phi = \text{const.}$, $\psi = \text{const.}$, where ϕ and ψ are the functions determined by equations (4) and (5), the form f becomes

$$f = \frac{1}{\mu^2 + \nu^3}\,(d\phi^2 + d\psi^2). \tag{6}$$

Hence, the system of lines $\phi = \text{const.}$, $\psi = \text{const.}$ is isothermal-conjugate. We remark that its determination depends upon the integration of the differential equation

$$\sqrt{D}\,du + \frac{D' + i\sqrt{DD'' - D'^2}}{\sqrt{D}}\,dv = 0. \tag{7}$$

Returning to the general coordinates u and v, we form the Beltrami differential parameters with respect to the form (1); they are

$$\Delta\theta = \frac{D\left(\frac{\partial\theta}{\partial v}\right)^2 - 2D'\frac{\partial\theta}{\partial v}\,\frac{\partial\theta}{\partial u} + D''\left(\frac{\partial\theta}{\partial u}\right)^2}{DD'' - D'^2}, \tag{8}$$

$$\Delta(\phi, \theta) = \frac{D\frac{\partial\theta}{\partial v}\,\frac{\partial\phi}{\partial v} - D'\left(\frac{\partial\theta}{\partial u}\,\frac{\partial\phi}{\partial v} + \frac{\partial\theta}{\partial v}\,\frac{\partial\phi}{\partial u}\right) + D''\frac{\partial\theta}{\partial u}\,\frac{\partial\phi}{\partial u}}{DD'' - D'^2}, \tag{9}$$

$$\Delta_2\theta = \frac{1}{\sqrt{DD'' - D'^2}}\left[\frac{\partial}{\partial u}\left(\frac{D''\frac{\partial\theta}{\partial u} - D'\frac{\partial\theta}{\partial v}}{\sqrt{DD'' - D'^2}}\right) + \frac{\partial}{\partial v}\left(\frac{D\frac{\partial\theta}{\partial v} - D'\frac{\partial\theta}{\partial u}}{\sqrt{DD'' - D'^2}}\right)\right]. \tag{10}$$

Let the surface be referred to the new system of lines $\phi=$ const., $\psi=$ const.; then

$$f=D_1\,d\phi^2+2D_1'\,d\phi\,d\psi+D_1''\,d\psi^2, \tag{11}$$

or, this may be written,

$$f=\frac{\Delta\psi\,d\phi^2-2\Delta(\phi,\psi)\,d\phi\,d\psi+\Delta\phi\,d\psi^2}{\Delta\phi\,.\,\Delta\psi-\Delta^2(\phi,\psi)}, \tag{12}$$

where, in consequence of their invariant character, the differential parameters may be formed either with respect to the form (1) or (11). Assuming that they are as given by (8), (9), (10), we remark that if the new variables ϕ and ψ are chosen in such a way that

$$\Delta\phi=\Delta\psi,\quad \Delta(\phi,\psi)=0, \tag{13}$$

the curves $\phi=$ const., $\psi=$ const. will form an isothermal-conjugate system, and conversely.

Let $\phi(u,v)$ be a particular solution of the partial differential equation

$$\Delta_2\theta=0, \tag{14}$$

that is,

$$\frac{\partial}{\partial u}\left(\frac{D''\dfrac{\partial\phi}{\partial u}-D'\dfrac{\partial\phi}{\partial v}}{\sqrt{DD''-D'^2}}\right)+\frac{\partial}{\partial v}\left(\frac{D\dfrac{\partial\phi}{\partial v}-D'\dfrac{\partial\phi}{\partial u}}{\sqrt{DD''-D'^2}}\right)=0. \tag{15}$$

In consequence of this we see that

$$-\frac{D\dfrac{\partial\phi}{\partial v}-D'\dfrac{\partial\phi}{\partial u}}{\sqrt{DD''-D'^2}}\,du+\frac{D''\dfrac{\partial\phi}{\partial u}-D'\dfrac{\partial\phi}{\partial v}}{\sqrt{DD''-D'^2}}\,dv$$

is an exact differential. Denote the latter by $d\psi$, then

$$\frac{\partial\psi}{\partial u}=-\frac{D\dfrac{\partial\phi}{\partial v}-D'\dfrac{\partial\phi}{\partial u}}{\sqrt{DD''-D'^2}},\quad \frac{\partial\psi}{\partial v}=\frac{D''\dfrac{\partial\phi}{\partial u}-D'\dfrac{\partial\phi}{\partial v}}{\sqrt{DD''-D'^2}}. \tag{16}$$

Solving for $\dfrac{\partial\phi}{\partial u}$, $\dfrac{\partial\phi}{\partial v}$, we have

$$\frac{\partial\phi}{\partial u}=\frac{D\dfrac{\partial\psi}{\partial v}-D'\dfrac{\partial\psi}{\partial u}}{\sqrt{DD''-D'^2}}\quad \frac{\partial\phi}{\partial v}=-\frac{D''\dfrac{\partial\psi}{\partial u}-D'\dfrac{\partial\psi}{\partial v}}{\sqrt{DD''-D'^2}}. \tag{17}$$

From this it follows that the function ψ, determined by quadratures from (16), is also a solution of equation (14). Hence, when one solution of this equation is known, another is given by quadratures. Two such solutions will be called *conjugate to one another.* It follows at once from (16) and (17) that ϕ and ψ satisfy the conditions

$$\Delta\phi = \Delta\psi, \quad \Delta(\phi, \psi) = 0.$$

Recalling the preceding results, we have that, *if ϕ and ψ are conjugate solutions of the equation*

$$\Delta_2\theta = 0,$$

the curves $\phi =$ const., $\psi =$ const. form an isothermal-conjugate system.

If the u and v lines form an isothermal-conjugate system, equations (17) reduce to

$$\frac{\partial\phi}{\partial u} = \frac{\partial\psi}{\partial v}, \quad \frac{\partial\phi}{\partial v} = -\frac{\partial\psi}{\partial u},$$

that is, $\phi + i\psi$ is a function of $u + iv$. Since the form (6) is not changed when ψ is replaced by $-\psi$, we have the theorem:

When an isothermal-conjugate system (ϕ, ψ) for a surface of positive curvature is known, every other isothermal-conjugate system (ϕ', ψ') can be obtained by putting

$$\phi' + i\psi' = F(\phi \pm i\psi),$$

where F is an arbitrary function.

Suppose that for a given conjugate system of lines $u =$ const., $v =$ const., we have

$$\frac{D}{D''} = \frac{U}{V}, \tag{18}$$

where U is a function of u alone and V is a function of v alone. Equation (18) can be replaced by

$$D = \lambda U, \quad D'' = \lambda V,$$

where λ is a function of u and v. We have now

$$f = \lambda(U du^2 + V dv^2). \tag{19}$$

If we introduce new variables u_1 and v_1, defined by

$$u_1 = \int\sqrt{U}du, \quad v_1 = \int\sqrt{V}dv, \tag{20}$$

the above form becomes

$$f = \lambda(du_1^2 + dv_1^2). \tag{19'}$$

Hence, the lines $u_1 = \text{const.}$, $v_1 = \text{const.}$ form an isothermal-conjugate system. But, from (20), it follows that this is the same system as $u = \text{const.}$, $v = \text{const.}$ Therefore, when the condition (18) is satisfied by a conjugate system of lines, the system is isothermal-conjugate, and conversely.

From the above it follows that if the curves $\phi = \text{const.}$ and their conjugates form an isothermal-conjugate system, it is possible to find a function F such that $F(\phi)$ is a particular solution of the equation

$$\Delta_2 \theta = 0.$$

Conversely, if $F(\phi)$ satisfies this equation, the curves $\phi = \text{const.}$ and their conjugate trajectories form an isothermal-conjugate system.

From (10) we have that

$$\Delta_2 [F(\phi)] = F'(\phi) \Delta_2 \phi + F''(\phi) \Delta\phi = 0,$$

where accents denote differentiation with respect to ϕ. Hence,

$$\frac{\Delta_2 \phi}{\Delta\phi} = -\frac{F''(\phi)}{F'(\phi)}. \tag{21}$$

Since the right-hand member is a function of ϕ alone, the left-hand member is a function of ϕ alone. Conversely, if $\frac{\Delta_2 \phi}{\Delta\phi}$ is a function of ϕ alone, $F(\phi)$ can be so determined that

$$\Delta_2 [F(\phi)] = 0.$$

We have then the following theorem: *The necessary and sufficient condition that a family of lines $\phi = \text{const.}$ and their conjugates form an isothermal-conjugate system, is that the ratio of the second and first differential parameters, formed with respect to the second fundamental quadratic form of the surface, is a function of ϕ alone.*

The conjugate trajectories of the curves $\phi = \text{const.}$ are found by the quadratures

$$\frac{\partial\psi}{\partial u} = -F'(\phi)\frac{D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}}{\sqrt{DD'' - D'^2}}, \quad \frac{\partial\psi}{\partial v} = F'(\phi)\frac{D''\frac{\partial\phi}{\partial u} - D'\frac{\partial\phi}{\partial v}}{\sqrt{DD'' - D'^2}}, \tag{22}$$

where

$$F'(\phi) = e^{-\int \frac{\Delta_2 \phi}{\Delta\phi} d\phi}.$$

It follows from this that *if the lines $\phi = \text{const.}$ belong to a double isothermal-conju-*

gate system, their conjugate trajectories are given by the differential equation

$$\frac{D\frac{\partial\phi}{\partial v}-D'\frac{\partial\phi}{\partial u}}{\sqrt{DD''-D'^2}}\,du-\frac{D''\frac{\partial\phi}{\partial u}-D'\frac{\partial\phi}{\partial v}}{\sqrt{DD''-D'^2}}\,dv=0,$$

where $e^{-\int\frac{\Delta_2\phi}{\Delta\phi}d\phi}$ *is an integrating factor.*

Suppose now that the family $\phi=\text{const.}$ of a double isothermal-conjugate system of lines is given by a differential equation of the first order,

$$M du+N dv=0;$$

then the other family of lines is given by

$$\frac{DN-D'M}{\sqrt{DD''-D'^2}}\,du+\frac{D'N-D''M}{\sqrt{DD''-D'^2}}\,dv=0;$$

From what precedes we know that these two equations have the same integrating factor; let it be denoted by μ and for the sake of brevity write

$$M_1=\frac{DN-D'M}{\sqrt{DD''-D'^2}},\quad N_1=\frac{D'N-D''M}{\sqrt{DD''-D'^2}}.$$

Then we find

$$\frac{\partial\log\mu}{\partial u}=\frac{M_1\left(\frac{\partial N}{\partial u}-\frac{\partial M}{\partial v}\right)-M\left(\frac{\partial N_1}{\partial u}-\frac{\partial M_1}{\partial v}\right)}{MN_1-M_1N}\equiv A,$$

$$\frac{\partial\log\mu}{\partial v}=\frac{N_1\left(\frac{\partial N}{\partial u}-\frac{\partial M}{\partial v}\right)-N\left(\frac{\partial N_1}{\partial u}-\frac{\partial M_1}{\partial v}\right)}{MN_1-M_1N}\equiv B,$$

from which μ can be obtained by quadratures. Hence, we have the following theorem analogous to that due to Lie concerning isothermal systems:

When one family of a double isothermal-conjugate system of lines on a surface of positive curvature is given by a differential equation of the first order, viz.,

$$M du+N dv=0,$$

the equation of the lines in finite terms can be obtained by quadratures.

From equations (4) and (5) it follows that when $\phi=\text{const.}$, $\psi=\text{const.}$ form an isothermal-conjugate system, the asymptotic lines are given by

$$\phi+i\psi=\text{const.},\quad \phi-i\psi=\text{const.}$$

Consider now surfaces whose total curvature is negative. The problem of finding an isothermal-conjugate system of lines on such a surface is the same as that of finding the asymptotic lines. For if u, v are the parameters referring to the asymptotic lines, the quadratic form f is simply

$$f = D' \, du \, dv \, .$$

Let now ϕ and ψ be two functions of u and v defined by the equations

$$\phi = f_1(u) + f_2(v) , \quad \psi = f_1(u) - f_2(v) , \tag{23}$$

where f_1 and f_2 are arbitrary functions. When the surface is referred to the system of lines $\phi =$ const., $\psi =$ const., we have

$$f = \lambda \, (d\phi^2 - d\psi^2) ,$$

where λ is a function of ϕ and ψ. Hence, *if the lines* $u =$ *const.*, $v =$ *const. are asymptotic lines, an isothermal-conjugate system of lines is given by*

$$f_1(u) + f_2(v) = \text{const.}, \quad f_1(u) - f_2(v) = \text{const.}$$

Conversely, if $\phi =$ *const.*, $\psi =$ *const. form an isothermal-conjugate system, then the asymptotic lines are given by*

$$F_1(\phi + \psi) = \text{const.}, \quad F_2(\phi - \psi) = \text{const.}$$

Let the surface be referred to an isothermal-conjugate system (u, v); then

$$D = - D'' , \quad D' = 0 \, .$$

Let X, Y, Z denote the direction cosines of the normal to the surface, and write

$$\mathcal{E} = \Sigma \left(\frac{\partial X}{\partial u} \right)^2 , \quad \mathcal{F} = \Sigma \frac{\partial X}{\partial u} \, \frac{\partial X}{\partial v} , \quad \mathcal{G} = \Sigma \left(\frac{\partial X}{\partial v} \right)^2 ,$$

then $$d\sigma^2 = \mathcal{E} \, du^2 + 2 \mathcal{F} \, du \, dv + \mathcal{G} \, dv^2$$

is the square of the linear element of the spherical representation of the surface. Denoting by $\begin{Bmatrix} rs \\ t \end{Bmatrix}'$ the Christoffel symbols formed with respect to this quadratic form, and writing

$$\rho = \frac{D}{\sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}} = \frac{- D''}{\sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}} ,$$

we have for the Codazzi equations, referring to the spherical representation of the surface,*

$$\frac{\partial \log \rho}{\partial v} = \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' - \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}', \quad \frac{\partial \log \rho}{\partial u} = \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}' - \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}', \tag{24}$$

and if (x, y, z) are the cartesian coordinates of a point on the surface, their derivatives with respect to u and v are given by

$$\left.\begin{aligned} \frac{\partial x}{\partial u} &= \frac{\rho}{\sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}} \left(- \mathcal{G} \frac{\partial X}{\partial u} + \mathcal{F} \frac{\partial X}{\partial v} \right), \\ \frac{\partial x}{\partial v} &= - \frac{\rho}{\sqrt{\mathcal{E}\mathcal{G} - \mathcal{F}^2}} \left(\mathcal{F} \frac{\partial X}{\partial u} - \mathcal{E} \frac{\partial X}{\partial v} \right), \end{aligned}\right\} \tag{25}$$

and analogous expressions in y and z.

From (24) it follows that *the necessary and sufficient condition that a system of u, v lines on a sphere be the spherical representation of an isothermal-conjugate system of lines on a surface of negative curvature is expressed by*

$$\frac{\partial}{\partial u} \left(\begin{Bmatrix} 11 \\ 2 \end{Bmatrix}' - \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}' \right) = \frac{\partial}{\partial v} \left(\begin{Bmatrix} 22 \\ 1 \end{Bmatrix}' - \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}' \right). \tag{26}$$

If we put

$$\sqrt{\rho}\, X = \xi, \quad \sqrt{\rho}\, Y = \eta, \quad \sqrt{\rho}\, Z = \zeta, \tag{27}$$

equations (25) may be replaced by

$$\frac{\partial x}{\partial u} = + \begin{vmatrix} \eta, & \zeta \\ \frac{\partial \eta}{\partial v}, & \frac{\partial \zeta}{\partial v} \end{vmatrix}, \quad \frac{\partial x}{\partial v} = + \begin{vmatrix} \eta, & \zeta \\ \frac{\partial \eta}{\partial u}, & \frac{\partial \zeta}{\partial u} \end{vmatrix}, \tag{28}$$

and analogous ones in y and z. It is readily shown that X, Y, Z are particular solutions of the equation

$$\frac{\partial^2 \phi}{\partial u^2} - \frac{\partial^2 \phi}{\partial v^2} + \frac{\partial \log \rho}{\partial u} \frac{\partial \phi}{\partial u} - \frac{\partial \log \rho}{\partial v} \frac{\partial \phi}{\partial v} + (\mathcal{E} - \mathcal{G}) \phi = 0.$$

Effecting the transformation

$$\sqrt{\rho}\, \phi = \theta,$$

we can bring this equation to the form

$$\frac{\partial^2 \theta}{\partial u^2} - \frac{\partial^2 \theta}{\partial v^2} = M\theta, \quad M = \frac{1}{\sqrt{\rho}} \left(\frac{\partial^2 \sqrt{\rho}}{\partial u^2} - \frac{\partial^2 \sqrt{\rho}}{\partial v^2} \right) - (\mathcal{E} - \mathcal{G}); \tag{29}$$

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* Bianchi, Lezioni, p. 131.

from (27) it follows that ξ, η, ζ are particular solutions of this equation. Conversely, if we take any three particular solutions ξ, η, ζ of an equation of the form (29) where M is any function of u and v, the surface, whose cartesian coordinates are determined from these solutions by means of (28), will have the u, v lines for an isothermal-conjugate system.

Recalling the expressions of Gauss for the second derivatives of the cartesian coordinates of a point on a surface,* we find that when the surface is one of negative curvature and is referred to an isothermal-conjugate system of lines, these coordinates must be simultaneous solutions of the equations

$$\left.\begin{aligned} \frac{\partial^2\theta}{\partial u\,\partial v} &= \begin{Bmatrix}12\\1\end{Bmatrix}\frac{\partial\theta}{\partial u} + \begin{Bmatrix}12\\2\end{Bmatrix}\frac{\partial\theta}{\partial v}, \\ \frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} &= \left[\begin{Bmatrix}11\\1\end{Bmatrix} + \begin{Bmatrix}22\\1\end{Bmatrix}\right]\frac{\partial\theta}{\partial u} + \left[\begin{Bmatrix}11\\2\end{Bmatrix} + \begin{Bmatrix}22\\2\end{Bmatrix}\right]\frac{\partial\theta}{\partial v}, \end{aligned}\right\} \tag{30}$$

where the Christoffel symbols $\begin{Bmatrix}rs\\t\end{Bmatrix}$ are formed with respect to the square of the linear element of the surface.

Conversely, if x, y, z are three linearly independent solutions of a completely integrable system of the form

$$\left.\begin{aligned} \frac{\partial^2\theta}{\partial u\,\partial v} &= a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \\ \frac{\partial^2\theta}{\partial u^2} + \frac{\partial^2\theta}{\partial v^2} &= \alpha\frac{\partial\theta}{\partial u} + \beta\frac{\partial\theta}{\partial v}, \end{aligned}\right\} \tag{31}$$

then x, y, z are the cartesian coordinates of a point on a surface of negative curvature referred to an isothermal-conjugate system of lines.

The results of the preceding sections are similar to those deduced by Bianchi† for surfaces of positive curvature referred to an isothermal-conjugate system; the various equations and conditions differ at most in sign.

It is readily seen that the plane is the only surface of zero curvature which admits isothermal-conjugate lines as defined. For, in this case,

$$DD'' - D'^2 = 0,$$

and hence only when

$$D = D' = D'' = 0,$$

can the above condition and that for isothermal-conjugate lines be satisfied simultaneously.

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* Bianchi, p. 88.

† Lezioni, pp. 133, 134.

If the distance from the point $(u+du,\ v+dv)$ on a surface to the tangent plane to the surface at the point $(u,\ v)$ be denoted by p, it can be shown that

$$p=\tfrac{1}{2}\left(D\,du^2+2D'\,du\,dv+D''\,dv^2\right)+\ \ldots\ldots,$$

where the unwritten terms are of a higher order than the second. Neglecting the latter, we find that the distance from $(u+du,\ dv)$ to this plane is $\frac{1}{2}D du^2$ and from $(u,\ v+dv)$ is $\frac{1}{2}D''dv^2$. If, then, we take $du=dv$, we find that when u and v refer to an isothermal-conjugate system, these distances are equal; moreover, they are measured in the same direction at points of positive curvature and in opposite directions at points of negative curvature.

Consider now, in connection with a surface S, whose u and v lines form an isothermal-conjugate system, the surface S_1 which corresponds to S by reciprocal radii vectores. Then, if $x_1,\ y_1,\ z_1$ denote the cartesian coordinates of a point on S_1, we have

$$x=\frac{\kappa^2x_1}{x_1^2+y_1^2+z_1^2},\quad y=\frac{\kappa^2y_1}{x_1^2+y_1^2+z_1^2},\quad z=\frac{\kappa^2z_1}{x_1^2+y_1^2+z_1^2}. \tag{32}$$

From the geometrical interpretation of isothermal-conjugate lines, as given above, and from the well-known fact that the transformation (32) changes a plane into a plane, a conjugate system of lines on S into a conjugate system on S_1, preserves angles and the ratios of infinitely small lengths, it follows that on S_1 the system of u and v lines form an isothermal-conjugate system. And since Darboux has shown that the lines of curvature on S_1 correspond to the lines of curvature on S, we have the theorem:

The transforms by reciprocal radii vectores of surfaces whose lines of curvature form an isothermal-conjugate system are also surfaces of this kind.

II.

Consider a surface S referred to a general system of curvilinear coordinates $(u,\ v)$; the square of its linear element takes the form

$$ds^2=E\,du^2+2F\,du\,dv+G\,dv^2. \tag{1}$$

The condition that the coordinate lines be the lines of curvature is

$$F=0,\quad D'=0,$$

and that these lines form an isothermal-conjugate system is, with proper choice of parameters,

$$D=\pm D'',$$

according as the surface is of positive or negative curvature.

Combining with the results found in the previous section with respect to surfaces of negative curvature, the similar results found by Bianchi for surfaces of positive curvature, we have that, in order that the lines of curvature of a surface may form an isothermal-conjugate system, it is necessary that the cartesian coordinates of its points satisfy simultaneously the two equations

$$\left.\begin{aligned} \frac{\partial^2\theta}{\partial u\,\partial v} &= \frac{\partial \log\sqrt{E}}{\partial v}\,\frac{\partial\theta}{\partial u} + \frac{\partial \log\sqrt{G}}{\partial u}\,\frac{\partial\theta}{\partial v}, \\ \frac{\partial^2\theta}{\partial u^2} \mp \frac{\partial^2\theta}{\partial v^2} &= \frac{1}{2E}\left(\frac{\partial E}{\partial u} \pm \frac{\partial G}{\partial u}\right)\frac{\partial\theta}{\partial u} \mp \left(\frac{\partial E}{\partial v} \pm \frac{\partial G}{\partial v}\right)\frac{\partial\theta}{\partial v}, \end{aligned}\right\} \tag{2}$$

where the upper sign holds for surfaces of positive curvature, and the lower for surfaces of negative curvature. Conversely, if two simultaneous equations of the form

$$\left.\begin{aligned} \frac{\partial^2\theta}{\partial u\,\partial v} &= a\,\frac{\partial\theta}{\partial u} + b\,\frac{\partial\theta}{\partial v}, \\ \frac{\partial^2\theta}{\partial u^2} \mp \frac{\partial^2\theta}{\partial v^2} &= \alpha\,\frac{\partial\theta}{\partial u} + \beta\,\frac{\partial\theta}{\partial v}, \end{aligned}\right\} \tag{3}$$

constitute a completely integrable system, and x, y, z are linearly independent solutions, satisfying the conditions

$$\frac{\partial x}{\partial u}\,\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\,\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\,\frac{\partial z}{\partial v} = 0,$$

then the locus of the point (x, y, z) is a surface of positive or negative curvature (according to the signs in the second equation (3)), whose lines of curvature form an isothermal-conjugate system.

It is evident that the problem of finding all the surfaces whose lines of curvature form an isothermal-conjugate system is a very difficult one, especially from the preceding point of view. We shall, however, by different methods find quite a number of such surfaces, but before proceeding to this, we will show that the solution of our problem depends upon the integration of a differential equation of the fourth order, similar to that found by Darboux for isothermic surfaces.

To this end we consider the surface referred to the system of tangential coordinates, imagined by Bonnet, for which the equation of the tangent plane takes the form

$$(\alpha + \beta)\,x + i\,(\beta - \alpha)\,y + (\alpha\beta - 1)\,z + \xi = 0. \tag{4}$$

Then the lines of curvature are given by

$$r\,d\alpha^2 - t\,d\beta^2 = 0,^* \tag{5}$$

and the z coordinate of the point of contact has the expression

$$z = \frac{\xi - p\alpha - q\beta}{1 + \alpha\beta}, \tag{6}$$

where we write

$$p = \frac{\partial\xi}{\partial\alpha}, \quad q = \frac{\partial\xi}{\partial\beta}, \quad r = \frac{\partial^2\xi}{\partial\alpha^2}, \quad s = \frac{\partial^2\xi}{\partial\alpha\,\partial\beta}, \quad t = \frac{\partial^2\xi}{\partial\beta^2}. \tag{7}$$

The second fundamental form is

$$f = \{(1 + \alpha\beta)(r\,d\alpha^2 + 2s\,d\alpha\,d\beta + t\,d\beta^2) + 2d\alpha\,d\beta\,(\xi - p\alpha - q\beta)\}\varkappa, \tag{8}$$

where $\varkappa$ is a function of α and β whose form is not essential.

If the parameters of the lines of curvature are u, v, then

$$du = \lambda(\sqrt{r}\,d\alpha - \sqrt{t}\,d\beta), \quad dv = \mu(\sqrt{r}\,d\alpha + \sqrt{t}\,d\beta), \tag{9}$$

where λ and μ are such that the expressions on the right are exact differentials. If the lines of curvature of the surface S form an isothermal-conjugate system,

$$f = D(du^2 + dv^2), \tag{10}$$

where S is a surface of positive curvature, as we will assume it is for the present. Replace du, dv in (10) by their expressions (9) and compare with (8); this gives

$$\left.\begin{aligned} &Dr(\lambda^2 + \mu^2) = \varkappa(1 + \alpha\beta)r, \quad Dt(\lambda^2 + \mu^2) = \varkappa(1 + \alpha\beta)t, \\ &D\sqrt{rt}(\mu^2 - \lambda^2) = \varkappa[s(1 + \alpha\beta) + \xi - p\alpha - q\beta] = \varkappa(1 + \alpha\beta)(s + z). \end{aligned}\right\} \tag{11}$$

Since the first two of these conditions are the same, we have two equations in λ^2 and μ^2; solving, we get

$$2D\lambda^2 = -\frac{s + z - \sqrt{rt}}{\sqrt{rt}}(1 + \alpha\beta)\varkappa, \quad 2D\mu^2 = \frac{s + z + \sqrt{rt}}{\sqrt{rt}}(1 + \alpha\beta)\varkappa. \tag{12}$$

Put

$$2D = \frac{(s + z)^2 - rt}{\theta^2\sqrt{rt}}(1 + \alpha\beta)\varkappa, \tag{13}$$

then

$$\lambda = \frac{i\theta}{(s + z + \sqrt{rt})^{\frac{1}{2}}}, \quad \mu = \frac{\theta}{(s + z - \sqrt{rt})^{\frac{1}{2}}}.$$

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* **Darboux, Leçons, I, p. 246.**

Hence θ must be such that

$$\theta\,\frac{\sqrt{r}\,d\alpha-\sqrt{t}\,d\beta}{(s+z+\sqrt{rt})^{\frac{1}{2}}},\qquad \theta\,\frac{\sqrt{r}\,d\alpha+\sqrt{t}\,d\beta}{(s+z-\sqrt{rt})^{\frac{1}{2}}}$$

are exact differentials. The condition of integrability of the first is

$$\sqrt{r}\,\frac{\partial\log\theta}{\partial\beta}+\sqrt{t}\,\frac{\partial\log\theta}{\partial\alpha}+\frac{\partial\sqrt{r}}{\partial\beta}+\frac{\partial\sqrt{t}}{\partial\alpha}-\sqrt{r}\,\frac{\partial}{\partial\beta}\log(z+s+\sqrt{rt})^{\frac{1}{2}}$$
$$-\sqrt{t}\,\frac{\partial}{\partial\alpha}\log(z+s+\sqrt{rt})^{\frac{1}{2}}=0.$$

The condition for the second is gotten from this by replacing $\sqrt{t}$ by $-\sqrt{t}$. From these two equations of condition we obtain the following:

$$2d\log\frac{\theta}{[(z+s)^2-rt]^{\frac{1}{4}}}=-\frac{\partial\log t}{\partial\alpha}\,d\alpha-\frac{\partial\log r}{\partial\beta}\,d\beta$$
$$+\sqrt{\frac{r}{t}}\,\frac{\partial}{\partial\beta}\log\left(\frac{z+s+\sqrt{rt}}{z+s-\sqrt{rt}}\right)^{\frac{1}{2}}d\alpha$$
$$+\sqrt{\frac{t}{r}}\,\frac{\partial}{\partial\alpha}\log\left(\frac{s+z+\sqrt{rt}}{s+z-\sqrt{rt}}\right)^{\frac{1}{2}}d\beta.$$

Expressing the condition that the right-hand member is an exact differential, we find the following equation of the fourth order which ξ must satisfy in order that the lines of curvature of S may form an isothermal-conjugate system :

$$\frac{\partial^2}{\partial\alpha\,\partial\beta}\log\frac{r}{t}-\frac{\partial}{\partial\alpha}\left[\sqrt{\frac{t}{r}}\,\frac{\partial}{\partial\alpha}\log\left(\frac{s+z+\sqrt{rt}}{s+z-\sqrt{rt}}\right)^{\frac{1}{2}}\right]$$
$$+\frac{\partial}{\partial\beta}\left[\sqrt{\frac{r}{t}}\,\frac{\partial}{\partial\beta}\log\left(\frac{s+z+\sqrt{rt}}{s+z-\sqrt{rt}}\right)^{\frac{1}{2}}\right]=0. \quad (14)$$

When S is a surface of negative curvature, we find, by methods similar to the preceding, that for the lines of curvature to form an isothormal-conjugate system we must have

$$Dr(\lambda^2-\mu^2)=(1+\alpha\beta)r,\qquad Dt(\lambda^2-\mu^2)=(1+\alpha\beta)t,$$
$$-D\sqrt{rt}(\mu^2+\lambda^2)=(1+\alpha\beta)(s+z),$$

from which, after making use of (13), we have

$$\lambda=\frac{i\theta}{(s+z+\sqrt{rt})^{\frac{1}{2}}},\quad \mu=\frac{i\theta}{(s+z-\sqrt{rt})^{\frac{1}{2}}}.$$

Proceeding as before, we are led to the same equation (14). Hence, when the function ξ for a surface of positive or negative curvature satisfies equation (14), the lines of curvature, which can be determined by quadratures by means of (9), form an isothermal-conjugate system.

The expression
$$\frac{s+z+\sqrt{rt}}{s+z-\sqrt{rt}}$$
is the ratio of the two radii of principal curvature of the surface.* Hence, since for a minimal surface,†
$$\frac{r}{t}=\frac{\phi(\alpha)}{f(\beta)},$$
it follows that the above equation is satisfied by the function ξ corresponding to minimal surfaces.

This result can be obtained also as follows: Denote by ρ_1 and ρ_2 the principal radii of curvature at a point, and let the surface be referred to its lines of curvature; then
$$\frac{1}{\rho_1}=\frac{D}{E},\quad \frac{1}{\rho_2}=\frac{D''}{G}.$$
If, now, S is a minimal surface, it is isothermic, and hence the parameters of the lines of curvature can be so chosen that $E=G$, and hence
$$D=-D''.$$
In a similar manner consider a sphere. Since an orthogonal system upon a sphere is at the same time conjugate and since the two fundamental forms are in constant ratio, every isothermal system upon the sphere is an isothermal-conjugate system and consequently the sphere can be looked upon in an infinity of ways as a surface with its lines of curvature forming an isothermal-conjugate system. Monge‡ has shown that the sphere is the only real surface whose principal radii of curvature are equal and of the same sign, hence minimal surfaces and sphere are the only real surfaces to be investigated by this method.

Recalling the notation of the previous section, we have the following relations, when a surface is referred to its lines of curvature :§
$$\mathfrak{E}=\frac{D^2}{E},\quad \mathfrak{G}=\frac{D''^2}{G}.$$

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* Darboux, Leçons, I, p. 246.

† Ib., p. 298.

‡ "Application de l'analyse à la géométrie," 5me ed., pp. 196-211.

§ Bianchi, Lezioni, p. 131.

From this it follows that when S is an isothermic surface and the spherical representation of its lines of curvature is also isothermal, then the lines of curvature of S form an isothermal-conjugate system. By means of this we shall be able to determine what surfaces of constant mean curvature have isothermal-conjugate lines of curvature. Bonnet* was the first to show that such surfaces are isothermic.

Willgrod† shows that for surfaces with constant mean curvature different from zero, the parameters of the lines of curvature can be so chosen that the following relations hold:

$$\mathfrak{E}\rho_1^2 = \mathfrak{G}\rho_2^2 = \frac{\rho_1 + \rho_2}{\rho_1 - \rho_2}.$$

From this it follows that for the spherical representation of S to be isothermal we must have

$$\frac{\rho_1}{\rho_2} = \frac{U}{V},$$

where U is a function of u alone and V is a function of v alone. But ρ_1 and ρ also satisfy the condition

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{c},$$

where c is a constant. From these two equations we have

$$\rho_1 = \frac{c(U+V)}{V}, \quad \rho_2 = \frac{c(U+V)}{U}.$$

When these expressions are substituted in the above equation, we get for $\mathfrak{E}$ and $\mathfrak{G}$

$$\mathfrak{E} = \frac{V^2}{c^2(U^2 - V^2)}, \quad \mathfrak{G} = \frac{U^2}{c^2(U^2 - V^2)}.$$

From these forms it is seen that in this case the orthogonal system upon the sphere is such that by a suitable choice of parameters the linear element can be given the form

$$d\sigma^2 = \frac{1}{(U_1 + V_1)}(du_1^2 + dv_1^2).$$

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* "Mémoire sur la theorie des surfaces applicables sur une surface donnée" (Journal de l'Ècole Polytechnique, XLII, Cahier, p. 77, 1867).

† Dissertation p. 30.

In order to determine the form of the functions U_1 and V_1, we note that the Gauss equation for the sphere referred to any isothermal orthogonal system, namely:*

$$\frac{\partial^2 \log \lambda}{du^2} + \frac{\partial^2 \log \lambda}{\partial v^2} + 2\lambda = 0,$$

where

$$\lambda = \mathfrak{E} = \mathfrak{G}$$

takes in this case the form

$$(U_1'' + V_1'' - 2)(U_1 + V_1) - U_1'^2 - V_1'^2 = 0,$$

where the accents denote differentiation with respect to u_1 and v_1 respectively. For the present we exclude the case where either U_1 or V_1 is a constant and we put

$$U_2 = U_1'^2, \quad V_2 = V_1'^2,$$

and take U_1 and V_1 for the independent variables. If we make this substitution and denote now by accents differentiation with respect to these new variables, the above equation becomes

$$(U_2' + V_2' - 4)(U_1 + V_1) - U_2 - V_2 = 0.$$

Differentiating this equation with respect to U_1 and then with respect to V_1, we are brought to the equation

$$U_2'' + V_2'' = 0.$$

From this it follows that U_2'' and V_2'' are constants differing only in sign. Hence

$$U_2 = \alpha U_1^2 + 2\beta U_1 + \gamma, \quad V_2 = -\alpha V_1^2 + 2b V_1 + c,$$

where $\alpha, \beta, \gamma, b, c$ are constants. When these values are substituted in the above equation, it becomes

$$U_1(2 + \beta - b) + V_1(2 - \beta + b) + \gamma + c = 0.$$

From the form of this equation, it follows that there cannot be a linear element of the form

$$d\sigma^2 = \frac{1}{(U_1 + V_1)}(du_1^2 + dv_1^2),$$

with neither U_1 nor V_1 constant. We shall consider now this exceptional case and assume that V_1 is constant. In this case the curves on the sphere are great circles with a common diameter and their orthogonal trajectories.

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* Bianchi, Lezioni, p. 67.

With respect to the corresponding surface, we have only to note that by retracing the steps which led to the linear element of the sphere in the above form, we find that when V_1 is a constant, V also is a constant. Then both ρ_1 and ρ_2 are functions of u alone, and consequently the surfaces are surfaces of revolution. From the well-known theorem of Bonnet with regard to the relation between surfaces of constant mean curvature and those with constant total curvature, one has that the surfaces of revolution of constant mean curvature are parallels of the surfaces of revolution with constant positive total curvature. Since the latter are known, the same is true of the former. Combining these results with those previously established with respect to the sphere and minimal surfaces, we have the theorem :

The only surfaces of constant mean curvature whose lines of curvature are isothermal-conjugate are the sphere, minimal surfaces and the surfaces of revolution whose mean curvature is constant.

We will deduce now the differential relation which exists between the principal radii of curvature of a surface whose lines of curvature are isothermal-conjugate. To this end we consider the Codazzi equations and the equation of Gauss, which, when the surface is referred to its lines of curvature reduce to*

$$\frac{\partial}{\partial v}\left(\frac{D}{\sqrt{E}}\right)-\frac{D''}{G}\,\frac{\partial\sqrt{E}}{\partial v}=0, \quad \frac{\partial}{\partial u}\left(\frac{D''}{\sqrt{G}}\right)-\frac{D}{E}\,\frac{\partial\sqrt{G}}{\partial u}=0, \tag{15}$$

$$\frac{DD''}{\sqrt{EG}}+\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{E}}\,\frac{\partial\sqrt{G}}{\partial u}\right)+\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{G}}\,\frac{\partial\sqrt{E}}{\partial v}\right)=0. \tag{16}$$

Now

$$E=\rho_1 D, \quad G=\rho_2 D'',$$

hence, equations (15) may be written

$$\left.\begin{aligned}\frac{\partial \log\sqrt{D}}{\partial v}&=\frac{\rho_2+\rho_1}{\rho_2-\rho_1}\,\frac{\partial\log\sqrt{\rho_1}}{\partial v},\\ \frac{\partial \log\sqrt{D''}}{\partial u}&=-\frac{\rho_2+\rho_1}{\rho_2-\rho_1}\,\frac{\partial\log\sqrt{\rho_2}}{\partial u}.\end{aligned}\right\} \tag{17}$$

Expressing the condition that the lines of curvature are isothermal-conjugate, that is,

$$\frac{D}{D''}=\frac{U}{V},$$

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* Bianchi, p. 93.

we have the following condition which ρ_1 and ρ_2 must satisfy:

$$\frac{\partial}{\partial u}\left(\frac{\rho_1+\rho_2}{\rho_1-\rho_2}\frac{\partial}{\partial v}\log\sqrt{\rho_1}\right)+\frac{\partial}{\partial v}\left(\frac{\rho_1+\rho_2}{\rho_1-\rho_2}\frac{\partial}{\partial u}\log\sqrt{\rho_2}\right)=0,$$

or, on developing,

$$\frac{1}{\rho_1}\frac{\partial^2\rho_1}{\partial u\,\partial v}+\frac{1}{\rho_2}\frac{\partial^2\rho_2}{\partial u\,\partial v}-\frac{\rho_1^2+2\rho_1\rho_2-\rho_2^2}{\rho_1^2(\rho_1^2-\rho_2^2)}\frac{\partial\rho_1}{\partial u}\frac{\partial\rho_1}{\partial v}$$
$$+\frac{\rho_2^2+2\rho_1\rho_2-\rho_1^2}{\rho_2^2(\rho_1^2-\rho_2^2)}\frac{\partial\rho_2}{\partial u}\frac{\partial\rho_2}{\partial v}=0, \tag{18}$$

which is the relation sought. If we put

$$\phi=\frac{1}{\rho_1\rho_2},\quad \psi=\frac{\rho_2-\rho_1}{\rho_1\rho_2(\rho_1+\rho_2)}, \tag{19}$$

then the above equation takes the simple form

$$2\psi\frac{\partial^2\phi}{\partial u\,\partial v}-\frac{\partial\phi}{\partial u}\frac{\partial\psi}{\partial v}-\frac{\partial\phi}{\partial v}\frac{\partial\psi}{\partial u}=0, \tag{20}$$

and the Codazzi equations become

$$\left.\begin{aligned}\psi\frac{\partial}{\partial v}\log D+\frac{\partial\phi}{\partial v}+\frac{\partial\psi}{\partial v}&=0,\\ \psi\frac{\partial}{\partial u}\log D-\frac{\partial\phi}{\partial u}+\frac{\partial\psi}{\partial u}&=0.\end{aligned}\right\} \tag{21}$$

From these equations we find

$$D=\frac{1}{\psi}e^{\int\frac{1}{\psi}\left(\frac{\partial\phi}{\partial u}du-\frac{\partial\phi}{\partial v}dv\right)}. \tag{22}$$

Making use of equations (15) and putting

$$D=\varepsilon D''.$$

where ε is $+1$ or -1, according as S is a surface of positive or negative curvature, we can put equation (16) in the form

$$\frac{D}{\sqrt{\varepsilon\rho_1\rho_2}}+\frac{\partial}{\partial u}\left[\sqrt{\varepsilon\frac{\rho_1}{\rho_2}}\frac{\rho_2}{\rho_2-\rho_1}\frac{\partial}{\partial u}\log\frac{1}{\rho_2}\right]$$
$$+\frac{\partial}{\partial v}\left[\sqrt{\varepsilon\frac{\rho_2}{\rho_1}}\frac{\rho_1}{\rho_1-\rho_2}\frac{\partial}{\partial v}\log\frac{1}{\rho_1}\right]=0. \tag{23}$$

III.

Consider the quadric surface whose equation is

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1. \tag{1}$$

The cartesian coordinates have the following expressions in terms of parameters referring to the lines of curvature :*

$$\left.\begin{aligned} x &=\sqrt{\frac{a\,(a-u)(a-v)}{(a-b)(a-c)}}\,, \\ y &=\sqrt{\frac{b\,(b-u)(b-v)}{(b-a)(b-c)}}\,, \\ z &=\sqrt{\frac{c\,(c-u)(c-v)}{(c-a)(c-b)}}\,. \end{aligned}\right\} \tag{2}$$

Calculating the direction cosines of the tangent plane to the surface at the point (x, y, z) and the coefficients of the first and second fundamental forms, we get

$$X=\sqrt{\frac{bc}{uv}}\sqrt{\frac{(a+u)(a+v)}{(a-b)(a-c)}}, \quad Y=\sqrt{\frac{ca}{uv}}\sqrt{\frac{(b+u)(b+v)}{(b-c)(b-a)}},$$

$$Z=\sqrt{\frac{ab}{uv}}\sqrt{\frac{(c+u)(c+v)}{(c-a)(c-b)}}. \tag{3}$$

$$E=\frac{u\,(u-v)}{4\,(a+u)(b+u)(c+u)}, \quad G=\frac{v\,(u-v)}{4\,(a+v)(b+v)(c+v)}, \tag{4}$$

$$D=-\tfrac{1}{4}\sqrt{\frac{abc}{uv}}\;\frac{u-v}{(a+u)(b+u)(c+u)},$$

$$D''=-\tfrac{1}{4}\sqrt{\frac{abc}{uv}}\;\frac{u-v}{(a+v)(b+v)(c+v)}. \tag{5}$$

From (4) we remark that the surfaces of the second degree whose equation is of the form (1) are isothermic and the expressions (5) show that *the lines of curvature of quadric surfaces are isothermal-conjugate.*

The surfaces of the second degree whose center is at an infinite distance can be looked upon as limiting cases of the preceding, and from this point of view we see that their lines of curvature are isothermal-conjugate. But we can prove it directly from the expressions for the cartesian coordinates of such surfaces. Thus, if the equation of the surface is

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* Darboux, Leçons, t. I, p. 156.

$$ax^2 + by^2 + 2cz + d = 0,$$

we have for the coordinates,*

$$\left.\begin{aligned} x &= c\sqrt{\frac{(a-b)}{b\,(ub+a)(vb+a)}}, \\ y &= c\sqrt{\frac{b-a}{a\,(ub+a)(vb+a)}}, \\ z &= \frac{c\,(a-b)}{2a}\,\frac{b^2uv-a^2}{(ub+a)(vb+a)} - \frac{d}{2c}. \end{aligned}\right\} \qquad (6)$$

Proceeding as in the former case, we find

$$X,\ Y,\ Z = \frac{a^2\sqrt{\frac{a-b}{a}},\quad ib^2\sqrt{\frac{a-b}{b}}\sqrt{uv},\quad \sqrt{ab\,(ub+a)(vb+a)}}{\sqrt{(b^2u+a^2)(b^2v+a^2)}}, \qquad (7)$$

$$E = \tfrac{1}{4}\,\frac{c^2\,(a-b)(u-v)\left(b^2+\frac{a^2}{u}\right)}{(ub+a)^4\,(vb+a)}, \quad G = -\tfrac{1}{4}\,\frac{c^2(a-b)(u-v)\left(b^2+\frac{a^2}{v}\right)}{(ub+a)(vb+a)^3}, \qquad (8)$$

$$D = \tfrac{1}{4}\,\frac{(ab)^{\frac{3}{2}}\,c\,(a-b)(u-v)}{\sqrt{(b^2u+a^2)(b^2v+a^2)(bu+a)(bv+a)}}\cdot\frac{1}{u\,(ub+a)^2}, \qquad (9)$$

$$D'' = \tfrac{1}{4}\,\frac{(ab)^{\frac{3}{2}}\,c\,(a-b)(u-v)}{\sqrt{(b^2u+a^2)(b^2v+a^2)(bu+a)(bv+a)}}\cdot\frac{1}{v\,(vb+a)^2}.$$

From these forms we see that *the lines of curvature of quadrics with their centers at an infinite distance are isothermal and isothermal-conjugate.*

Surfaces of revolution can be defined by the equations

$$x = r\cos v, \quad y = r\sin v, \quad z = \phi(r).$$

Then the square of the linear element takes the form

$$ds^2 = [1 + \phi'(r)^2]\,dr^2 + r^2\,dv^2,$$

and it is readily found that the second fundamental form is

$$f = \frac{\phi''(r)\,dr^2 - r\phi'(r)\,dv^2}{\sqrt{1+\phi'(r)^2}}.$$

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* Willgrod, ib., p. 27.

Hence, *surfaces of revolution have isothermal-conjugate lines of curvature*, and, moreover, the ratio of D and D'' is a function of r alone.

IV.

In this section we wish to determine those surfaces which, together with their parallels, have isothermal-conjugate lines of curvature.

In solving this problem, we make use of the equation

$$\frac{1}{\rho_1}\;\frac{\partial^2\rho_1}{\partial u\,\partial v}+\frac{1}{\rho_2}\;\frac{\partial^2\rho_2}{\partial u\,\partial v}-\frac{\rho_1^2+2\rho_1\rho_2-\rho_2^2}{\rho_1^2(\rho_1^2-\rho_2^2)}\;\frac{\partial\rho_1}{\partial u}\;\frac{\partial\rho_1}{\partial v}$$
$$+\frac{\rho_2^2+2\rho_1\rho_2-\rho_1^2}{\rho_2^2(\rho_1^2-\rho_2^2)}\;\frac{\partial\rho_2}{\partial u}\;\frac{\partial\rho_2}{\partial v}=0, \qquad (1)$$

which, as we have seen, is the condition which the principal radii of curvature of a surface S must satisfy in order that its lines of curvature be isothermal-conjugate. We have then that this equation must also be satisfied by ρ_1+a and ρ_2+a ($a=\text{const.}$), for all values of a, in order that the lines of curvature on the parallels of S may form an isothermal system. Since this condition must be satisfied identically for values of a, it is only necessary to substitute ρ_1+a, ρ_2+a for ρ_1 and ρ_2 respectively and equate to zero the coefficients of the different powers of a. This gives

$$\frac{\partial^2\rho_1}{\partial u\,\partial v}+\frac{\partial^2\rho_2}{\partial u\,\partial v}-\frac{1}{\rho_1-\rho_2}\left(\frac{\partial\rho_1}{\partial u}\;\frac{\partial\rho_1}{\partial v}-\frac{\partial\rho_2}{\partial u}\;\frac{\partial\rho_2}{\partial v}\right)=0,$$

$$(3\rho_1+5\rho_2)\frac{\partial^2\rho_1}{\partial u\,\partial v}+(5\rho_1+3\rho_2)\frac{\partial^2\rho_2}{\partial u\,\partial v}-\frac{4\rho_1+\rho_2}{\rho_1-\rho_2}\left(\frac{\partial\rho_1}{\partial u}\;\frac{\partial\rho_1}{\partial v}-\frac{\partial\rho_2}{\partial u}\;\frac{\partial\rho_2}{\partial v}\right)=0,$$

$$(\rho_1^2+4\rho_2^2+7\rho_1\rho_2)\frac{\partial^2\rho_1}{\partial u\,\partial v}+(\rho_2^2+4\rho_1^2+7\rho_1\rho_2)\;\frac{\partial^2\rho_2}{\partial u\,\partial v}$$
$$-\frac{\rho_1^2+10\rho_1\rho_2+\rho_2^2}{\rho_1-\rho_2}\left(\frac{\partial\rho_1}{\partial u}\;\frac{\partial\rho_1}{\partial v}-\frac{\partial\rho_2}{\partial u}\;\frac{\partial\rho_2}{\partial v}\right)=0.$$

$$\rho_2(2\rho_1^2+5\rho_1\rho_2+\rho_2^2)\;\frac{\partial^2\rho_1}{\partial u\,\partial v}+\rho_1(\rho_1^2+5\rho_1\rho_2+2\rho_2^2)\frac{\partial^2\rho_2}{\partial u\,\partial v}$$
$$-\frac{2}{\rho_1-\rho_2}\left[\rho_2(4\rho_1\rho_2+\rho_1^2-\rho_2^2)\frac{\partial\rho_1}{\partial u}\;\frac{\partial\rho_1}{\partial v}-\rho_1(4\rho_1\rho_2+\rho_2^2-\rho_1^2)\frac{\partial\rho_2}{\partial u}\;\frac{\partial\rho_2}{\partial v}\right]=0,$$

$$\frac{1}{\rho_1}\;\frac{\partial^2\rho_1}{\partial u\,\partial v}+\frac{1}{\rho_2}\;\frac{\partial^2\rho_2}{\partial u\,\partial v}-\frac{1}{\rho_1^2\rho_2^2(\rho_1^2-\rho_2^2)}\left[\rho_2^2(\rho_1^2+2\rho_1\rho_2-\rho_2^2)\frac{\partial\rho_1}{\partial u}\;\frac{\partial\rho_1}{\partial v}\right.$$
$$\left.-\rho_1^2(\rho_2^2+2\rho_1\rho_2-\rho_1^2)\frac{\partial\rho_2}{\partial u}\;\frac{\partial\rho_2}{\partial v}\right]=0.$$

Combining the first two of the above equations, we find that either

$$\frac{\partial^2\rho_1}{\partial u\,\partial v} - \frac{\partial^2\rho_2}{\partial u\,\partial v} = 0,$$

or

$$\rho_1 - \rho_2 = 0.$$

The latter simply gives the case of the sphere, which is an evident solution of the problem. Taking the former and proceeding step by step with the above equations, we are brought to the following equations of condition, which must be satisfied by surfaces furnishing a solution of the problem:

$$\frac{\partial^2\rho_1}{\partial u\,\partial v} = 0, \quad \frac{\partial^2\rho_2}{\partial u\,\partial v} = 0, \quad \frac{\partial\rho_1}{\partial u}\,\frac{\partial\rho_1}{\partial v} = 0, \quad \frac{\partial\rho_2}{\partial u}\,\frac{\partial\rho_2}{\partial v} = 0. \qquad (2)$$

There are four possible solutions of this system of equations; they are

$$\left.\begin{array}{lll} 1^\circ. & \rho_1 = f(u), & \rho_2 = \phi(v), \\ 2^\circ. & \rho_1 = f(v), & \rho_2 = \phi(u), \\ 3^\circ. & \rho_1 = f(u), & \rho_2 = \phi(u), \\ 4^\circ. & \rho_1 = f(v), & \rho_2 = \phi(v). \end{array}\right\} \qquad (3)$$

Since the last two cases are practically identical, we have only the first three to consider. We have found that for surfaces with isothermal-conjugate lines of curvature the following relation exists:

$$\frac{D}{\sqrt{\varepsilon\rho_1\rho_2}} + \frac{\partial}{\partial u}\left[\sqrt{\varepsilon\frac{\rho_1}{\rho_2}}\;\frac{\rho_2}{\rho_2-\rho_1}\;\frac{\partial}{\partial u}\log\frac{1}{\rho_2}\right] + \frac{\partial}{\partial v}\left[\sqrt{\frac{\varepsilon\rho_2}{\rho_1}}\;\frac{\rho_1}{\rho_1-\rho_2}\;\frac{\partial}{\partial v}\log\frac{1}{\rho_1}\right] = 0. \qquad (4)$$

$$1^\circ. \quad \rho_1 = f(u), \qquad \rho_2 = \phi(v).$$

From (4) it is seen that either

$$\rho_1 = \rho_2 = \text{const.},$$

or

$$D = 0.$$

Hence the sphere and the plane are solutions of the problem.

$$2^\circ. \quad \rho_1 = f(v), \qquad \rho_2 = \phi(u).$$

Since, by definition, ρ_1 is the radius of curvature of the normal section tangent to the curves $v =$ const., we see that for this class of surfaces ρ_1 is constant along such a curve. Hence, the normals to the surface along such a curve meet in a point, and, consequently, form a cone, and the curve lies on the sphere whose center is at the vertex of the cone and of radius ρ_1. It is evident, therefore, that the surface is the envelope of a family of spheres of radius $\rho_1 = f(v)$. In a similar way it can be shown that the surface is the envelope of a family of spheres of radius $\phi(u)$. Hence, the surfaces of the case under discussion are *cyclides of Dupin.* The coefficients of the second fundamental form of any surface of this class can be found at once from (4),

$$3^\circ. \quad \rho_1 = f(u), \qquad \rho_2 = \phi(u).$$

The surfaces corresponding to these expressions are evidently surfaces of revolution; from (4) we find

$$D = \varepsilon D'' = F(u),$$

where ε is $+1$ or -1, according as S has positive or negative curvature.

Hence, *the only surfaces which, together with their parallels, have isothermal-conjugate lines of curvature, are the plane, sphere, cyclides of Dupin and surfaces of revolution.*

Combining (3) and

$$\rho_1 = \frac{E}{D}, \quad \rho_2 = \frac{G}{D''},$$

we see that since $D'' = \varepsilon D$,

$$\frac{E}{G} = \frac{U}{V},$$

hence all of these surfaces are isothermic. Conversely, from the results found by Willgrod in solving the similar problem for isothermic surfaces, we remark that all surfaces which, together with their parallels are isothermic, have isothermal-conjugate lines of curvature and their parallels have the same property.

V.

We will determine the surfaces whose principal radii satisfy a relation and whose lines of curvature form an isothermal-conjugate system.

In solving this problem, we make use of the following equation, which we have found must be satisfied in order that the lines of curvature of the corresponding surface be isothermal-conjugate:

$$2\psi \frac{\partial^2 \phi}{\partial u \partial v} - \frac{\partial \phi}{\partial u} \frac{\partial \psi}{\partial v} - \frac{\partial \phi}{\partial v} \frac{\partial \psi}{\partial u} = 0, \tag{1}$$

where
$$\phi = \frac{1}{\rho_1 \rho_2}, \quad \psi = \frac{\rho_2 - \rho_1}{\rho_1 \rho_2 (\rho_1 + \rho_2)}. \tag{2}$$

If ρ_1 is a function of ρ_2, then ψ will be a function of ϕ except when the relation between the radii is

$$\frac{1}{\rho_1 \rho_2} = \text{const.} \quad \text{or} \quad \frac{\rho_2 - \rho_1}{\rho_1 \rho_2 (\rho_1 + \rho_2)} = \text{const.}$$

From (1) it is seen that the former of these exceptional cases gives a solution of our problem. In order that the second may give a solution, we must have

$$\phi = \frac{1}{\rho_1 \rho_2} = U + V,$$

where U is a function of u alone and V is a function of v alone. From (II, 22) it follows that

$$D = \frac{1}{C} e^{\frac{U - V}{C}}.$$

The functions U and V must be of such a character as to satisfy the transformed Gauss equation

$$\frac{D}{\sqrt{\epsilon \rho_1 \rho_2}} + \frac{\partial}{\partial u}\left[\sqrt{\frac{\epsilon \rho_1}{\rho_2}} \frac{\rho_2}{\rho_2 - \rho_1} \frac{\partial}{\partial u} \log \frac{1}{\rho_2}\right]$$
$$+ \frac{\partial}{\partial v}\left[\sqrt{\frac{\epsilon \rho_2}{\rho_1}} \frac{\rho_1}{\rho_1 - \rho_2} \frac{\partial}{\partial v} \log \frac{1}{\rho_1}\right] = 0. \tag{3}$$

Proceed now to the general case where ψ is a function of ϕ; then

$$\frac{\partial \psi}{\partial u} = \frac{\partial \psi}{\partial \phi} \frac{\partial \phi}{\partial u}, \quad \frac{\partial \psi}{\partial v} = \frac{\partial \psi}{\partial \phi} \frac{\partial \phi}{\partial v}.$$

Hence, equation (1) reduces to

$$\psi \frac{\partial^2 \phi}{\partial u \partial v} - \frac{\partial \psi}{\partial \phi} \frac{\partial \phi}{\partial u} \frac{\partial \phi}{\partial v} = 0,$$

or
$$\frac{\partial}{\partial v}\left(\frac{\frac{\partial \phi}{\partial u}}{\psi}\right) = 0, \quad \frac{\partial}{\partial u}\left(\frac{\frac{\partial \phi}{\partial v}}{\psi}\right) = 0,$$

If, now, we write
$$W = U + V,$$

where U is a function of u alone and V is a function of v alone, we have from the above equation

$$\frac{\partial \phi}{\partial u} = \psi \frac{\partial W}{\partial u}, \quad \frac{\partial \phi}{\partial v} = \psi \frac{\partial W}{\partial v}, \tag{4}$$

whence

$$W = \int \frac{d\phi}{\psi}.$$

Hence ϕ and ψ are functions of W; from (4) we see that these functions are such that ψ is the derivative of ϕ with respect to W. We can write therefore

$$\phi = \frac{1}{\rho_1 \rho_2} = f(W), \quad \psi = \frac{\rho_2 - \rho_1}{\rho_1 \rho_2 (\rho_1 + \rho_2)} = f'(W). \tag{5}$$

From (II, 22) we find

$$D = \frac{e^{U-V}}{f'(W)}. \tag{6}$$

The functions f, U, V must be such as to verify equation (3), which, on replacing ρ_1, ρ_2, D by their expressions from (5) and (6), becomes

$$\frac{e^{U-V}.\sqrt{f}}{f'} + \frac{\partial}{\partial u}\left[\frac{U'}{2}\sqrt{\frac{f-f'}{f+f'}}\left(\frac{f'}{f} + \frac{f'-f''}{f-f'} - \frac{f'+f''}{f+f'} + \frac{2(f'-f'')}{f'}\right)\right]$$
$$+ \frac{\partial}{\partial v}\left[\frac{V'}{2}\sqrt{\frac{f+f'}{f-f'}}\left(\frac{f'}{f} + \frac{f'+f''}{f+f'} - \frac{f'-f''}{f-f'} - \frac{2(f'+f'')}{f'}\right)\right] = 0. \tag{7}$$

It is evident that surfaces of revolution correspond to the case when ϕ and ψ are functions of only one of the parameters.

From the theorem of Weingarten,* we have that for surfaces whose principal radii satisfy a relation, the parameters (u, v) of the lines of curvature *can be so chosen* that the square of the linear element of the spherical representation of the surface takes the form

$$ds'^2 = \frac{du^2}{\alpha^2} + \frac{dv^2}{\theta'^2(\alpha)}, \tag{8}$$

where α is a function of u, v, and the character of the function θ depends upon the relation between the radii; this function θ being such that

$$\rho_1 = \theta(\alpha), \quad \rho_2 = \theta(\alpha) - \alpha\theta'(\alpha). \tag{9}$$

From this it is seen that by changing α we get different members of a family of surfaces whose radii of curvature satisfy the same relation. We wish to find whether there are any such families, whose lines of curvature are isothermal-conjugate and in such a way that for the parameters, as above chosen,

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* "Ueber die Oberflächen, für welche einer der beiden Hauptkrümmungshalbmesser eine Function des andern ist." Crelle, Vol. 62, pp. 160–173.

$$D = \varepsilon D''.$$

Consider first the case $D = D''$; it is readily shown that in consequence of this we have

$$\mathcal{E}\rho_1 = \mathcal{G}\rho_2.$$

Replacing the quantities in this equation by their expressions as given by (8) and (9), we get

$$\frac{\theta(\alpha)}{\alpha^2} = \frac{\theta(\alpha) - \alpha\theta'(\alpha)}{\theta'^2(\alpha)}.$$

Hence, the solution of our problem reduces to the integration of this equation It can readily be brought to Clairant's form, from which we get the genera solution

$$\theta^2(\alpha) = c\alpha^2 + c^2;$$

this gives

$$\rho_1 = c\sqrt{1 + \frac{\alpha^2}{c}}, \quad \rho_2 = \frac{c}{\sqrt{1 + \frac{\alpha^2}{c}}},$$

hence, by elimination, we find that the surfaces for which

$$\rho_1\rho_2 = c^2,$$

furnish the solution.

Proceeding in a similar manner for surfaces of negative curvature, we find

$$\rho_1 = c\sqrt{\frac{\alpha^2}{c} - 1}, \qquad \rho_2 = -\frac{c}{\sqrt{\frac{\alpha^2}{c} - 1}},$$

that is,

$$\rho_1\rho_2 = -c^2.$$

Hence, *the surfaces of constant curvature furnish the general solution of our problem.*

VI.

Consider a surface whose lines of curvature are plane in both systems, and let the planes of these lines have for equations

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1, \\ a_2x + b_2y + c_2z &= d_2, \end{aligned}\right\} \tag{1}$$

where a_1, b_1, c_1, d_1 are functions of u alone and a_2, b_2, c_2, d_2 are functions of v alone—u, v being parameters of the lines of curvature.

Differentiating these equations with respect to u, we have

$$\left.\begin{aligned} a_1\frac{\partial x}{\partial u}+b_1\frac{\partial y}{\partial u}+c_1\frac{\partial z}{\partial u}&=-\left(xa_1'+yb_1'+zc_1'-d_1'\right),\\ a_2\frac{\partial x}{\partial u}+b_2\frac{\partial y}{\partial u}+c_2\frac{\partial z}{\partial u}&=0,\end{aligned}\right\}\tag{2}$$

where accents denote differentiation with respect to u; in like manner,

$$\left.\begin{aligned} a_2\frac{\partial x}{\partial v}+b_2\frac{\partial y}{\partial v}+c_2\frac{\partial z}{\partial v}&=-\left(xa_2'+yb_2'+zc_2'-d_2'\right),\\ a_1\frac{\partial x}{\partial v}+b_1\frac{\partial y}{\partial v}+c_1\frac{\partial z}{\partial v}&=0,\end{aligned}\right\}\tag{3}$$

where the accents denote differentiation with respect to v. Combining equations (2) with

$$X\frac{\partial x}{\partial u}+Y\frac{\partial y}{\partial u}+Z\frac{\partial z}{\partial u}=0,\tag{4}$$

and solving for $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$, $\frac{\partial z}{\partial u}$, we find

$$\left.\begin{aligned} \frac{\partial x}{\partial u}&=-\frac{(xa_1'+yb_1'+zc_1'-d_1')(b_2Z-c_2Y)}{\Delta},\\ \frac{\partial y}{\partial u}&=-\frac{(xa_1'+yb_1'+zc_1'-d_1')(c_2X-a_2Z)}{\Delta},\\ \frac{\partial z}{\partial u}&=-\frac{(xa_1'+yb_1'+zc_1'-d_1')(a_2Y-b_2X)}{\Delta},\end{aligned}\right\}\tag{5}$$

where Δ denotes the determinant of the system of equations. From these expressions we find

$$\begin{aligned} D=-\Sigma\frac{\partial x}{\partial u}\,\frac{\partial X}{\partial u}=\frac{(xa_1'+yb_1'+zc_1'-d_1')}{\Delta}\Big[a_2\left(Y\frac{\partial Z}{\partial u}-Z\frac{\partial Y}{\partial u}\right)&\\ +b_2\left(Z\frac{\partial X}{\partial u}-X\frac{\partial Z}{\partial u}\right)+c_2\left(X\frac{\partial Y}{\partial u}-Y\frac{\partial X}{\partial u}\right)\Big].&\end{aligned}\tag{6}$$

By treating equations (3) and

$$X\frac{\partial x}{\partial v}+Y\frac{\partial y}{\partial v}+Z\frac{\partial z}{\partial v}=0$$

in a similar manner, we find

$$D'' = \frac{(xa_2' + yb_2' + zc_2' - d_2')}{\Delta}\Big[a_1\Big(Y\frac{\partial Z}{\partial v} - Z\frac{\partial Y}{\partial v}\Big) + b_1\Big(Z\frac{\partial X}{\partial v} - X\frac{\partial Z}{\partial v}\Big) + c_1\Big(X\frac{\partial Y}{\partial v} - Y\frac{\partial X}{\partial v}\Big)\Big]. \tag{7}$$

Hence the necessary and sufficient condition that the lines of curvature on such a surface form an isothermal-conjugate system is that the functions $a_1, a_2, \ldots, d_2$ satisfy the relation

$$\frac{(xa_1' + yb_1' + zc_1' - d_1')\Big[a_2\Big(Y\frac{\partial Z}{\partial u} - Z\frac{\partial Y}{\partial u}\Big) + b_2\Big(Z\frac{\partial X}{\partial u} - X\frac{\partial Z}{\partial u}\Big) + c_2\Big(X\frac{\partial Y}{\partial u} - Y\frac{\partial X}{\partial u}\Big)\Big]}{(xa_2' + yb_2' + zc_2' - d_2')\Big[a_1\Big(Y\frac{\partial Z}{\partial v} - Z\frac{\partial Y}{\partial v}\Big) + b_1\Big(Z\frac{\partial X}{\partial v} - X\frac{\partial Z}{\partial v}\Big) + c_1\Big(X\frac{\partial Y}{\partial v} - Y\frac{\partial X}{\partial v}\Big)\Big]} = \frac{U}{V}. \tag{8}$$

where U is a function of u alone and V is a function of v alone.

We will recall that all surfaces with plane lines of curvature in both systems can be divided into three families:*

1°. *Moulure surfaces* and *surfaces of revolution.* The surfaces of this family may be defined as the envelopes of the plane

$$uz - x\cos v - y\sin v = f(u) + \phi(v), \tag{9}$$

for different forms of the functions f and ϕ. It is manifest that the surfaces of revolution correspond to $\phi(v) = 0$;

2°. Surfaces defined as the envelope of the plane

$$ux - vy + (\lambda\sqrt{1-u^2} - \sqrt{\lambda^2 - 1}\,.\,\sqrt{1+v^2})z = f(u) - \phi(v); \tag{10}$$

3°. Surfaces defined as the envelope of the plane

$$2ux + 2vy + (1 - u^2 - v^2)z = f(u) + \phi(v). \tag{11}$$

Geometrically, the three families are distinguished by their spherical representation.† Those of the first family have for spherical representation a system of great circles pivoting about a diameter and their orthogonal parallels. The lines of curvature of the surfaces of the second family have for their spherical representation a double system of small circles whose respective planes meet in two reciprocal straight lines; when these lines are tangent to the sphere, the spherical representation is that of the third family. For a method of point and tangential generation of all surfaces with plane lines of curvature in both systems the reader is referred to Darboux.*

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* Darboux, Leçons, t. I, p. 130.

† Caronnet, Thesis, p. 10,

We will now discuss each of these families.

Surfaces of the First Family.

As previously remarked, these surfaces may be defined as the envelope of the plane

$$uz - x\cos v - y\sin v = f(u) + \phi(v). \tag{9}$$

On taking the derivatives of this equation with respect to u and v respectively, we get the equations of the planes of the lines $u=$ const. and $v=$ const.; they are

$$\left.\begin{aligned} &z = f'(u), \\ &x\sin v - y\cos v = \phi'(v). \end{aligned}\right\} \tag{12}$$

Comparing these equations with equations (1), we remark that for this case

$$\left.\begin{aligned} &a_1 = b_1 = 0, \quad c_1 = 1, \quad d_1 = f'(u), \\ &a_2 = \sin v, \quad b_2 = -\cos v, \quad c_2 = 0, \quad d_2 = \phi'(v); \end{aligned}\right\} \tag{13}$$

and from (9),

$$X = \frac{-\cos v}{\sqrt{1+u^2}}, \quad Y = -\frac{\sin v}{\sqrt{1+u^2}}, \quad Z = \frac{u}{\sqrt{1+u^2}}.$$

When these expressions are substituted in the conditional equation (8), it is found that $f(u)$ and $\phi(v)$ must be such that

$$\frac{f''(u)}{x\cos v + y\sin v - \phi''(v)} = \frac{U}{V},$$

which, in consequence of equations (9) and (12), can be written

$$\frac{1}{uf'(u) - f(u) - \phi(v) - \phi''(v)} = \frac{U_2}{V}, \tag{14}$$

where now $U_2 = \dfrac{U}{f''(u)}$. In order that this relation may be satisfied either

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* Leçons, t. I, p. 132.

1°. $uf'(u) - f(u) = C_1$ and $\phi(v)$ has any form whatever; or,

2°. $\phi''(v) + \phi(v) = C_2$ and $f(u)$ has any form, where C_1 and C_2 are constants.

In the former case, we would have

$$f(u) = C_3 u - C_1. \tag{15}$$

However, in this case,

$$z = f'(u) = C_3,$$

that is, the surface would be a plane curve, and hence this case does not hold.

In the second case,

$$\phi(v) = (c_1 + c_2)\cos v + i(c_1 - c_2)\sin v + C_2, \tag{16}$$

where c_1 and c_2 are arbitrary constants.

We have then for surfaces of the first family whose lines of curvature form an isothermal-conjugate system, *surfaces of revolution* and those *moulure surfaces* for which $\phi(v)$ has the form (16) and $f(u)$ has any form with the exception of (15).

Surfaces of the Second Family.

These surfaces being the envelopes of the plane

$$ux - vy + (\lambda\sqrt{1-u^2} - \sqrt{\lambda^2-1}\sqrt{1+v^2})z = f(u) - \phi(v), \tag{10}$$

the planes of their lines of curvature are

$$\left.\begin{aligned} x - \frac{\lambda u}{\sqrt{1-u^2}}z &= f'(u), \\ y + \frac{\sqrt{\lambda^2-1}}{\sqrt{1+v^2}}z &= \phi'(v). \end{aligned}\right\} \tag{17}$$

Comparing these with equations (1), we have for this case

$$\left.\begin{aligned} &a_1 = 1, \quad b_1 = 0, \quad c_1 = -\frac{\lambda u}{\sqrt{1-u^2}}, \quad d_1 = f'(u), \\ &a_2 = 0, \quad b_2 = 1, \quad c_2 = \frac{\sqrt{\lambda^2-1}\,.\,v}{\sqrt{1+v^2}}, \quad d_2 = \phi'(v), \end{aligned}\right\} \tag{18}$$

and from (10),

$$X,\ Y,\ Z = \frac{u,\ -v,\ \lambda\sqrt{1-u^2} - \sqrt{\lambda^2-1}\,.\,\sqrt{1+v^2}}{\left(u^2(1-\lambda^2) + \lambda^2 v^2 + 2\lambda^2 - 1 - 2\lambda\sqrt{\lambda^2-1}\sqrt{1-u^2}\,.\,\sqrt{1-v^2}\right)^{\frac{1}{2}}}. \tag{18'}$$

When these values are substituted in equation (8), the latter becomes

$$\frac{f''(u)+\lambda\left(\frac{u}{\sqrt{1-u^2}}\right)'z}{\phi''(v)-\sqrt{\lambda^2-1}\left(\frac{v}{\sqrt{1-v^2}}\right)'z}=\frac{U}{V}. \tag{19}$$

Replacing z by its expression, as determined from (10) and (17), viz.:

$$z=\frac{[f(u)-uf'(u)]-[\phi(v)-v\phi'(v)]}{\frac{\lambda}{\sqrt{1-u^2}}-\frac{\sqrt{\lambda^2-1}}{\sqrt{1+v^2}}},$$

we can put the equation in the form

$$\frac{\dfrac{f''(u)}{\lambda\left(\frac{u}{\sqrt{1-u^2}}\right)}\left(\frac{\lambda}{\sqrt{1-u^2}}-\frac{\sqrt{\lambda^2-1}}{\sqrt{1+v^2}}\right)+f(u)-uf'(u)-\phi(v)+v\phi'(v)}{\dfrac{\phi''(v)}{\sqrt{\lambda^2-1}.\left(\frac{v}{\sqrt{1+v^2}}\right)}\left(\frac{\lambda}{\sqrt{1-u^2}}-\frac{\sqrt{\lambda^2-1}}{\sqrt{1+v^2}}\right)-f(u)+uf'(u)+\phi(v)-v\phi'(v)}=\frac{U_1}{V_1}, \tag{20}$$

where U_1 and V_1 are new functions of u and v respectively.

But this equation, which determines the character of f and ϕ so that the lines of curvature on the corresponding surface forms an *isothermal-conjugate* system, is exactly the same equation which Caronnet* has found in seeking to determine what surfaces of the second family are *isothermic.* We have then only to recall his results in order to have an answer to our problem, at the same time referring the reader to the elegant method used by Caronnet for the solution of the above equation. They are as follows:

$$\left.\begin{aligned} f(u)&=l\sqrt{1-u^2}+mu+n,\\ \phi(v)&=l'\sqrt{1+v^2}+m'v+n',\end{aligned}\right\} \tag{I}$$

where l m, n, l', m', n' are any constants whatever. The surface corresponding to these forms of f and ϕ is the *cyclide of Dupin* of the *fourth* order.

$$\left.\begin{aligned} f(u)&=\lambda^2\left(1-\frac{u}{2}\log\frac{1+u}{1-u}\right),\\ \phi(v)&=(\lambda^2-1)(1+v\tan^{-1}v).\end{aligned}\right\} \tag{II}$$

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* These—"Recherches sur les surfaces isothermiques et les surfaces dont les rayons de courbure sont fonctions l'un de l'autre, p. 13.

$$\left.\begin{aligned} f(u) &= -\frac{2t}{1+t^2}\Big[\frac{t}{\lambda+c} - \frac{1}{(\lambda-c)t} - \frac{4\lambda^2}{(\lambda^2-c^2)^{\frac{3}{2}}}\tan^{-1}\sqrt{\frac{\lambda+c}{\lambda-c}}\,t\Big], \\ \phi(v) &= -\frac{2\tau}{1+\tau^2}\Big[\frac{\tau}{\sqrt{\lambda^2-1}+c} - \frac{1}{\sqrt{(\lambda^2-1-c)}\,\tau} \\ &\qquad - \frac{4(\lambda^2-1)}{(\lambda^2-1-c^2)^{\frac{3}{2}}}\tan^{-1}\sqrt{\frac{\sqrt{\lambda^2-1}+c}{\sqrt{\lambda^2-1}-c}}\,\tau\Big], \end{aligned}\right\} \quad \text{(III)}$$

where, in order to simplify the expressions, we have written

$$u = \frac{2t}{1+t^2}, \quad v = \frac{2i\tau}{1+\tau^2},$$

and c is an arbitrary constant. When, in particular $c = 0$, the functions define the *minimal surface of Bonnet*. In determining the above forms for f and ϕ, it was assumed that the product

$$(\lambda^2 - c^2)(\lambda^2 - 1 - c^2)$$

is different from zero. In discussing this special case, there is need only of considering that where $\lambda^2 - c^2 = 0$ on account of symmetry. This gives

$$\left.\begin{aligned} f(u) &= -\frac{3t^4+6t^2-1}{3\lambda t^2(1+t^2)}, \\ \phi(v) &= -\frac{2\tau}{1+\tau^2}\Big[\frac{\tau}{\sqrt{\lambda^2-1}+\lambda} - \frac{1}{(\sqrt{\lambda^2-1}-\lambda)\tau} \\ &\qquad - \frac{4(\lambda^2-1)}{i}\tan^{-1}\sqrt{\frac{\sqrt{\lambda^2-1}+\lambda}{\sqrt{\lambda^2-1}-\lambda}}\,\tau\Big]. \end{aligned}\right\} \quad \text{(III}'\text{)}$$

Finally

$$\left.\begin{aligned} f(u) &= \frac{\lambda^2\sqrt{\alpha^2-\lambda^2}}{\alpha}\int\frac{\sqrt{(\alpha^2+2c\alpha+c_1)(\alpha^2-\lambda^2)}}{(\alpha^2-\lambda^2)^2}\,d\alpha, \\ \phi(v) &= -\frac{(\lambda^2-1)\sqrt{\beta^2-\lambda^2+1}}{\beta}\int\frac{\sqrt{(\beta^2+2c\beta+c_1)(\beta^2-\lambda^2+1)}}{(\beta^2-\lambda^2+1)^2}\,d\beta, \end{aligned}\right\} \quad \text{(IV)}$$

where

$$\alpha = \frac{\lambda}{\sqrt{1-u^2}}, \quad \beta = \frac{\sqrt{\lambda^2-1}}{\sqrt{1+v^2}},$$

and c, c_1 are arbitrary constants.

Surfaces of the Third Family.

These are defined as the envelope of the plane

$$2ux + 2vy + (1 - u^2 - v^2)z = f(u) + \phi(v). \qquad (11)$$

The planes of the lines of curvature are

$$x - uz = \frac{f'(u)}{2}, \quad y - vz = \frac{\phi'(v)}{2}. \tag{21}$$

Reasoning in a manner analogous to that used with respect to the other two families, we find that f and ϕ must satisfy the identity

$$\frac{\frac{1}{2}f''(u) + z}{\frac{1}{2}\phi''(v) + z} = \frac{U}{V}, \tag{22}$$

which, after replacing z by its expression as found from (11) and (21), is

$$\frac{\frac{1}{2}(1 + u^2 + v^2)f''(u) + f(u) - uf'(u) + \phi(v) - v\phi'(v)}{\frac{1}{2}(1 + u^2 + v^2)\phi''(v) + f(u) - uf'(u) + \phi(v) - v\phi'(v)} = \frac{U}{V}. \tag{22'}$$

Again we remark that this is the same identity which Caronnet* has found in seeking for those surfaces of the third family which are *isothermic.* As before, we will simply recall his results; they are

$$\left.\begin{aligned} f(u) &= \lambda u^2 + mu + n, \\ \phi(v) &= \lambda' v^2 + m'v + n', \end{aligned}\right\} \tag{I}$$

where λ, m, n, λ', m', n' are arbitrary constants. These surfaces are *cyclides of Dupin* of the *third* order.

$$\left.\begin{aligned} f(u) &= -(3u^2 + u^4), \\ \phi(v) &= \quad 3v^2 + v^4. \end{aligned}\right\} \tag{II}$$

This is the *minimal surface of Enneper*, whose rectangular cartesian coordinates have the following expressions:

$$\begin{aligned} x &= 3u + 3uv^2 - u^3, \\ y &= 3v + 3vu^2 - v^3, \\ z &= 3u^2 - 3v^2. \end{aligned}$$

$$\left.\begin{aligned} f(u) &= (1 - c)^{-\frac{3}{2}} u \tan^{-1}\frac{u}{\sqrt{1 - c}} + \frac{1}{1 - c}, \\ \phi(v) &= c^{-\frac{3}{2}} v \tan^{-1}\frac{v}{\sqrt{c}} + \frac{1}{c}, \end{aligned}\right\} \tag{III}$$

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* These, p. 22.

where c is a constant.

$$\left.\begin{aligned} f(u) &= u \tan^{-1} u + 1, \\ \phi(v) &= \frac{1}{3v^2}. \end{aligned}\right\} \tag{IV}$$

$$\left.\begin{aligned} f(u) &= u \int \frac{\sqrt{(1+u^2)^2 + 2c(1+u^2) + c_1}}{u^2}\, du, \\ \phi(v) &= v \int \frac{\sqrt{v^4 - 2cv^2 + c_1}}{v_2}\, dv, \end{aligned}\right\} \tag{V}$$

where c and c_1 are constants.

It is of interest to remark that the condition (22) can be readily obtained from the general differential equation of the fourth order which ξ satisfies. For, on writing

$$2u = -(\alpha + \beta), \quad 2v = -i(\beta - \alpha), \quad \xi = f(u) + \phi(v),$$

equation (11) takes the form

$$(\alpha + \beta)x + i(\beta - \alpha)y + (\alpha\beta + 1)z + \xi = 0.$$

We have at once

$$p = \frac{\partial \xi}{\partial \alpha} = -\tfrac{1}{2}(f' - i\phi'), \quad q = \frac{\partial \xi}{\partial \beta} = -\tfrac{1}{2}(f' + i\phi'),$$

$$r = \frac{\partial^2 \xi}{\partial \alpha^2} = \tfrac{1}{4}(f'' - \phi''), \quad s = \frac{\partial^2 \xi}{\partial \alpha\, \partial \beta} = \tfrac{1}{4}(f'' + \phi''), \quad t = \frac{\partial^2 \xi}{\partial \beta^2} = \tfrac{1}{4}(f'' - \phi''),$$

hence
$$r = t, \quad s + r = \tfrac{1}{2} f'', \quad s - r = \tfrac{1}{2}\phi''.$$

Substituting these expressions in the equation for ξ, we find that f and ϕ must satisfy the equation

$$\frac{\partial^2}{\partial \alpha^2} \log \left(\frac{\frac{1}{2} f'' + z}{\frac{1}{2}\phi'' + z}\right)^{\frac{1}{2}} - \frac{\partial^2}{\partial \beta^2} \log \left(\frac{\frac{1}{2} f'' + z}{\frac{1}{2}\phi'' + z}\right)^{\frac{1}{2}} = 0,$$

or, by returning to variables u, v,

$$\frac{\partial^2}{\partial u\, \partial v} \log \left(\frac{\frac{1}{2} f'' + z}{\frac{1}{2}\phi'' + z}\right)^{\frac{1}{2}} = 0,$$

that is,
$$\frac{\frac{1}{2} f''(u) + z}{\frac{1}{2}\phi''(v) + z} = \frac{U}{V}.$$

We have shown that when the lines of curvature of a surface are isothermal and isothermal-conjugate, their spherical representation forms an isothermal system. Further, it is well known that the surface obtained from an isothermic surface by reciprocal radii vectores is isothermic. Hence, combining the results of the preceding sections with those known for isothermic surfaces, we have the result:

Quadrics, the sphere, minimal surfaces, surfaces of revolution, the cyclides of Dupin, the members of the second and third families of surfaces with plane lines of curvature in both systems whose coordinates have been given, together with the surfaces obtained from the preceding by reciprocal radii vectores, have an isothermal spherical representation of their lines of curvature.

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